NOTES ON LAPLACE TRNAFORM

1. Laplace transform

In this section, we first briefly discuss the Laplace transform of some functions and use to solve some ODEs as well as PDEs with constant coefficients. In the Laplace transform method, differential equations are transformed into algebraic equations and then the solutions of the differential equations are obtained by means of taking the inverse Laplace transform.

Definition 1.1. Let $f(t)$ be a function defined for $t \in [0, \infty)$. The integral

$$F(s) = \int_0^\infty e^{-st} f(t)\ dt \tag{1.1}$$

is called the Laplace transform of $f(t)$ provided the integral exists. We denote the Laplace transform of f by F. That is

$$L[f(t)] = F(s).$$

Then the function f is called the inverse Laplace transform of $f(t)$, we write

$$L^{-1}[F(s)] = f(t).$$

Example 1.1. Find the Laplace transform of $f(t) = k,\ \forall\ t \geq 0$, where k is a constant.

Solution: By the definition of the Laplace transform, we get

$$L[k] = \int_0^\infty k e^{-st} dt = -\frac{ke^{-st}}{s}\Big]_0^\infty = \frac{k}{s}.$$

Example 1.2. Find the Laplace transform of $f(t) = e^{bt},\ \forall\ t \geq 0$, where b is a constant.

Solution: By the definition of the Laplace transform, we get

$$L[e^{bt}] = \int_0^\infty e^{-st} e^{bt} dt = -\frac{ke^{-(s-b)t}}{s-b}\Big]_0^\infty = \frac{1}{s-b},\ s > b.$$

Theorem 1.1. ***(Linear property of Laplace transform)*** *Let f and g be any two functions whose Laplace transform exist. Then for any two constants a and b, we have*

$$L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)].$$

Proof. As the Laplace transform of f and g exist, so the Laplace transform of $af + bg$ exists. By the definition of the Laplace transform, we get

$$\begin{aligned} L[af(t) + bg(t)] &= \int_0^\infty [af(t) + bg(t)]e^{-st}dt \\ &= a\int_0^\infty f(t)e^{-st}dt + b\int_0^\infty g(t)e^{-st}g(t)dt \\ &= aL[f(t)] + bL[g(t)]. \end{aligned}$$

□

Definition 1.2. A function f defined on $[a, b]$ is said to be piecewise continuous if

(i) there are at most a finite number of points $t_0, t_1, t_2, ..., t_n$ with $t_{k-1} < t_k,\ k = 1, 2, ..., n;$

(ii) $\lim_{t\to t_k+} f(t),\ \lim_{t\to t_k-} f(t)$ exist;

(iii) f is continuous on each subinterval $t_{k-1} < t < t_k$.

The points t_k's are called the jump discontinuities of f.

For example $f(x) = [x],\ x \in [0, 4]$, where $[x]$ denotes the greatest integer function less than or equal to x, is piecewise continuous function on $[0, 4]$. The points of discontinuity of f are $1, 2, 3, 4$.

Definition 1.3. A function f is said to be of exponential order α, if there are positive constants α and M such that

$$|f(t)| \le Me^{\alpha t},\ t \ge 0.$$

If f is of exponential order α, then we have

$$\lim_{t\to\infty} |f(t)|e^{-\alpha t} \le M.$$

We know that

$$|t| \le e^t,\ e^{-4t} \le e^t,\ |\cos t| \le e^t,\ |\sin t| \le e^t.$$

Thus the functions $t, e^{-4t}, \cos t, \sin t$ are of exponential order. Again by L'Hospital's rule, we have

$$\lim_{t\to\infty} \frac{|t|^n}{e^{\alpha t}} = 0 \le M.$$

Thus t^n is of exponential order. However, the function e^{t^2} is not of exponential order as for any finite value of α. We have

$$\lim_{t\to\infty} \frac{e^{t^2}}{e^{\alpha t}} = \lim_{t\to\infty} e^{t(t-\alpha)} = \infty.$$

Theorem 1.2. ***(Sufficient condition for the existence of Laplace transform)*** *If f is piecewise continuous on $[0, \infty)$ and is of exponential order α, then the Laplace transform of f exists for $s > \alpha$.*

Proof. As f is piecewise continuous on $[0, \infty)$, so f is piecewise continuous on each closed and bounded subinterval $[0, T]$ of $[0, \infty)$. Then $e^{-st}f(t)$ is piecewise continuous $[0, T]$ and hence the integral $\int_0^T e^{-st}f(t)dt$ exists. Thus the existence of the Laplace transform depends on the convergence of the integral $\int_0^T e^{-st}f(t)dt$ as $T \to \infty$. Since f is of exponential oder α, so there exists $M > 0$ such that

$$|f(t)| \leq Me^{\alpha t}$$
$$\Rightarrow 0 \leq |f(t)|e^{-st} \leq Me^{\alpha t}e^{-st} \text{ for } \ t \geq 0.$$

Now

$$\begin{aligned}\int_0^T Me^{\alpha t}e^{-st}dt &= -\frac{M}{s-\alpha}[e^{-(s-\alpha)T} - 1] \\ &\to \frac{M}{s-\alpha} \text{ as } T \to \infty \text{ if } s > \alpha.\end{aligned}$$

By comparison test, the integral $\int_0^\infty |f(t)|e^{-st}dt$ converges. That is the integral $\int_0^\infty f(t)e^{-st}dt$ converges absolutely. Thus the integral $\int_0^\infty f(t)e^{-st}dt$ converges. That is the Laplace transform of f exists. □

Remark 1.1. The sufficient condition may not be necessary. That is there are functions which are not piecewise continuous on $[0, \infty)$ but the Laplace transform exist. For example, let $f(t) = t^{-\frac{1}{2}}$. Then

$$\begin{aligned}L[t^{-\frac{1}{2}}] &= \int_0^\infty t^{-\frac{1}{2}}e^{-st}dt \\ &= \int_0^\infty t^{\frac{1}{2}-1}e^{-st}dt \\ &= \frac{1}{\sqrt{s}}\int_0^\infty u^{\frac{1}{2}-1}e^{-u}du \text{ putting } \ u = st \\ &= \frac{1}{\sqrt{s}}\Gamma(\frac{1}{2}) \\ &= \frac{\sqrt{\pi}}{\sqrt{s}},\end{aligned}$$

where $\Gamma\alpha = \int_0^\infty u^{\alpha-1}e^{-u}du$ is the Gamma function for $\alpha \neq 0, -1, -2, ...$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Remark 1.2. If f satisfies the condition for the existence of Laplace transform and if $L(f) = F(s)$, then it follows from the proof of the above theorem that

(i) $\lim_{s\to\infty} F(s) = 0$ (ii) $\lim_{s\to\infty} sF(s)$ is bounded.

Example 1.3. Find the Laplace transform of t^α.

Solution: We have by definition

$$\begin{aligned} L[t^\alpha] &= \int_0^\infty t^\alpha e^{-st} dt \\ &= \frac{1}{s^{\alpha+1}} \int_0^\infty u^{\alpha+1-1} e^{-u} du \text{ putting } \ u = st \\ &= \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \end{aligned}$$

where $\alpha \neq -1, -2, -3, ...$

Example 1.4. Find the Laplace transform of $\cos wt$ and $\sin wt$.

Solution: We have by definition

$$\begin{aligned} L[\cos wt] &= \int_0^\infty \cos wt e^{-st} dt \\ &= \frac{1}{2} \int_0^\infty (e^{iwt} + e^{-iwt}) e^{-st} dt \\ &= \frac{1}{2} \left[\frac{-e^{-(s-iw)t}}{s-iw} + \frac{-e^{-(s+iw)t}}{s+iw} \right]_0^\infty \\ &= \frac{1}{2} \left[\frac{1}{s-iw} + \frac{1}{s+iw} \right] \\ &= \frac{s}{s^2+w^2} \end{aligned}$$

since $\lim_{t \to \infty} |e^{-(s-iw)t}| = 0$ and $\lim_{t \to \infty} |e^{-(s+iw)t}| = 0$. Continuing in the same way we can show that $L[\sin wt] = \frac{w}{s^2+w^2}$.

Example 1.5. Find the Laplace transform of $\cosh wt$ and $\sinh wt$.

Solution: We have by definition

$$\begin{aligned} L[\cosh wt] &= \int_0^\infty \cosh wt e^{-st} dt \\ &= \frac{1}{2}\int_0^\infty (e^{wt} + e^{-wt}) e^{-st} dt \\ &= \frac{1}{2}\left[\frac{-e^{-(s-w)t}}{s-w} + \frac{-e^{-(s+w)t}}{s+w}\right]_0^\infty \\ &= \frac{1}{2}\left[\frac{1}{s-w} + \frac{1}{s+w}\right] \\ &= \frac{s}{s^2 - w^2}. \end{aligned}$$

Continuing in the same way we can show that $L[\sinh wt] = \frac{w}{s^2-w^2}$.

1.1. Questions.

(1) Find the Laplace transform of the following functions
   (a) $\frac{1}{t^{3/2}}$.
   (b) $e^t \cosh t$.
   (c) $\sinh^2 t$.
   (d) $t^2 e^{2t}$.
   (e) $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & t \ge 1. \end{cases}$
   (f) $f(t) = \begin{cases} \cos t, & 0 \le t < \pi \\ 0, & t \ge \pi. \end{cases}$

(2) Find the inverse Laplace transform of the following
   (a) $\frac{1}{s^2+\pi^2}$.
   (b) $\frac{s+3}{(s-s)(s+2)}$.
   (c) $\frac{2s+6}{s^2+16}$.
   (d) $\frac{s^2+2s+5}{(s-1)(s-2)(s-3)}$.

1.2. Laplace Transform of Derivatives and Integrals.

Under the Laplace transformation, the derivatives a function f in t variables transforms to polynomial multiplication in s variable. This helps to solve the initial value problems for the ODEs with constant coefficients. On the other hand, under the Laplace transformation, the integral of a function f in t variable transforms to polynomial division in s variable, which is useful in solving the integral equations. We have the following theorem for the Laplace transform of derivatives.

Theorem 1.3. *Let f be of exponential order and be continuous function on $[0,\infty)$ such that f' is piecewise continuous on $[0,\infty)$ and is of exponential order. Then*

$$L[f'(t)] = sL[f(t)] - f(0).$$

Proof. First we assume that f' is continuous on $[0,\infty)$. Then by the definition of the Laplace transform, we get

$$\begin{aligned} L[f'(t)] &= \int_0^\infty f'(t)e^{-st}dt \\ &= f(t)e^{-st}]_0^\infty + s\int_0^\infty f(t)e^{-st}dt \\ &= -f(0) + sL[f]. \end{aligned}$$

Thus we establish the result assuming f' is continuous. Next we assume that f' is piecewise continuous and has a point of jump discontinuity at $t = T$ (say). Then

$$\begin{aligned} L[f'(t)] &= \int_0^\infty f'(t)e^{-st}dt \\ &= \lim_{\epsilon\to 0}\left[\int_0^{T-\epsilon} f'(t)e^{-st}dt + \int_{T+\epsilon}^\infty f'(t)e^{-st}dt\right] \\ &= \Bigg[e^{-st}f(t)]_0^{T-\epsilon} + e^{-st}f(t)]_{T+\epsilon}^\infty \\ &+ s\int_0^{T-\epsilon} f(t)e^{-st}dt + s\int_{T+\epsilon}^\infty f(t)e^{-st}dt\Bigg] \\ &= -f(0) + sL[f], \end{aligned}$$

since f is continuous, so $\lim_{\epsilon\to 0}[e^{-s(T-\epsilon)}f(T-\epsilon) - e^{-s(T+\epsilon)}f(T+\epsilon)] = 0$. □

Remark 1.3. Assume that $f, f', f'', f''', ..., f^{(n-1)}$ are continuous $[0,\infty)$ and are of exponential order with $f^{(n)}$ is at least piecewise continuous on $[0,\infty)$. Replacing f by f', we get

$$\begin{aligned} L[f''] &= sL[f'] - f'(0) \\ &= s[sL[f] - f(0)] - f'(0) \\ &= s^2L(f) - sf(0) - f'(0). \end{aligned}$$

Continuing in a similar way

$$L[f^{(n)}] = s^nL[f] - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0).$$

Example 1.6. Find the Laplace transform of $\sin^2 4t$.

Solution: We have $f(t) = \sin^2 4t$. Then $f(0) = 0$ and $f'(t) = 4\sin 8t$.

$$\begin{aligned} L[f'] &= -f(0) + sL[f] \\ \Rightarrow \frac{32}{s^2+8^2} &= sL[f] \\ \Rightarrow L[f] &= \frac{32}{s(s^2+64)}. \end{aligned}$$

Theorem 1.4. ***(Laplace transform of Integral)*** *If f is a piecewise continuous on $[0,\infty)$ and is of exponential order α, then*

$$L\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}L[f],\ s > 0,\ s > \alpha.$$

Proof. Define $g(t) = \int_0^t f(\tau)d\tau$. Then

$$|g(t)| \le \int_0^t |f(\tau)|d\tau \le \int_0^t Me^{\alpha\tau}d\tau = \frac{M}{\alpha}(e^{\alpha t} - 1) \le \frac{M}{\alpha}e^{\alpha t}.$$

That is g is of exponential order α. Also g is piecewise continuous on $[0,\infty)$. Further,

$$g'(t) = f(t)$$

except the points of discontinuity of f. Thus

$$\begin{aligned} L[g'] &= L[f] \\ \Rightarrow sL[g] - g(0) &= L[f] \\ \Rightarrow L[g] &= \frac{L[f]}{s} \\ \Rightarrow L\left[\int_0^t f(\tau)\right] &= \frac{L[f]}{s}. \end{aligned}$$

□

Example 1.7. Solve the following initial value problem

$$y'' + y = t,\ y(0) = 1,\ y'(0) = 0.$$

Solution: The given equation is

$$y'' + y = t. \tag{1.2}$$

Taking the Laplace transform on both sides of (1.2), we get

$$
\begin{aligned}
L[y''] + L[y] &= L[t] \\
\text{or } s^2 L[y] - sy(0) - y'(0) + L[y] &= \frac{1}{s^2} \\
\text{or } (s^2+1)L[y] &= \frac{1}{s^2} + s \\
\text{or } L[y] &= \frac{1}{s^2(s^2+1)} + \frac{s}{s^2+1} \\
\text{or } L[y] &= \frac{1}{s^2} - \frac{1}{(s^2+1)} + \frac{s}{s^2+1}.
\end{aligned}
$$

Taking the inverse Laplace transform, we get solution of the given problem

$$y(t) = t - \sin t + \cos t.$$

Example 1.8. Solve the following integro-differential equation

$$y' - y - 6\int_0^t y(\tau)d\tau = \sin t, \; y(0) = 2. \tag{1.3}$$

Solution: Taking the Laplace transform to both sides of (1.3), we get

$$
\begin{aligned}
& L[y'] - L[y] - 6\frac{L[y]}{s} = \frac{1}{s^2+1} \\
& \text{or } sL[y] - 2 - L[y] - 6\frac{L[y]}{s} = \frac{1}{s^2+1} \\
& \text{or } (s - 1 - \frac{6}{s})L[y] = \frac{1}{s^2+1} + 2 \\
& \text{or } L[y] = \frac{s}{(s^2+1)(s^2-s-6)} + \frac{2s}{s^2-s-6} \\
& \text{or } L[y] = -\frac{7}{50}\frac{s}{s^2+1} - \frac{1}{50}\frac{1}{s^2+1} + \frac{63}{50}\frac{1}{s-3} + \frac{44}{50}\frac{1}{s+2}.
\end{aligned}
$$

Taking the inverse Laplace transform, we get

$$y(t) = -\frac{7}{50}\cos t - \frac{1}{50}\sin t + \frac{63}{50}e^{-3t} + \frac{44}{50}e^{-2t}.$$

Example 1.9. Solve the following system of equations

$$
\begin{aligned}
y_1' + y_2 &= 2 - \sin t, \\
y_1 - y_2' &= t + \cos t, \; y_1(0) = 3, \; y_2(0) = 0.
\end{aligned}
$$

Solution: Taking Laplace transform to the above equations and using the initial values, we get

$$\begin{aligned} L[y_1'] + L[y_2] &= \frac{2}{s} - L[\sin t], \\ L[y_1] - L[y_2'] &= \frac{1}{s^2} + L[\cos t]. \end{aligned}$$

Or,

$$\begin{aligned} sL[y_1] - 3 + L[y_2] &= \frac{2}{s} - \frac{1}{s^2+1}, \\ L[y_1] - sL[y_2] &= \frac{1}{s^2} + \frac{s}{s^2+1}. \end{aligned}$$

Solving for $L[y_1]$ and $L[y_2]$ we get

$$\begin{aligned} L[y_1] &= \frac{1}{(s^2+1)} + \frac{3s}{(s^2+1)} + \frac{1}{s^2}, \\ L[y_2] &= \frac{1}{s(s^2+1)} - \frac{2}{(s^2+1)}. \end{aligned}$$

Taking the inverse Laplace transform, we get

$$\begin{aligned} y_1(t) &= \sin t + 3\cos t + t, \\ y_2(t) &= 1 - \cos t + 2\sin t, \end{aligned}$$

since $L^{-1}[\frac{1}{s(s^2+1)}] = L^{-1}[\frac{\frac{1}{(s^2+1)}}{s}] = \int_0^t \sin\tau d\tau = 1 - \cos t$.

1.3. **Questions.** Solve the following equations

(1) $2y" - y' - y = \cos t,\ y(0) = 1,\ y'(0) = 0.$
(2) $y'' + 5y + 4y = e^{3t},\ y(0) = 0,\ y'(0) = 3.$
(3) $y' - 4y + 3\int_0^t y(\tau)d\tau = t,\ y(0) = 1.$
(4) $y' + 6y + 5\int_0^t y(\tau)d\tau = 1 + t,\ y(0) = 1.$
(5)

$$\begin{aligned} y_1' - y_2 &= 1, \\ 4y_1 + y_2' &= t,\ y_1(0) = -1,\ y_2(0) = 1. \end{aligned}$$

(6)

$$\begin{aligned} y_1' - 3y_2 &= 3\sin(3t), \\ 3y_1 + y_2' &= 3(1 - \cos(3t))t,\ y_1(0) = -3,\ y_2(0) = 1. \end{aligned}$$

1.4. **Differentiation and Integration of the Laplace Transform.** We have discussed the Laplace transform method of solving the initial value problems for the ODEs with constant coefficients. However the methods discussed so far are not sufficient for solving the initial value problems with variable coefficients. The differentiation of the Laplace transform is useful to solve such problems. We note that the polynomial multiplication in t variable transforms under the Laplace transformation to differentiation in s variable. Again division by t transforms to integration in s variable. The following theorem gives the differentiation of the Laplace transform.

Theorem 1.5. ***(Differentiation of the Laplace transform)*** *Let f be a piecewise continuous function $[0, \infty)$ and be of exponential order. Then*

$$L[tf(t)] = -\frac{d}{ds}F(s) \text{ and } L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s),$$

where $F(s) = L[f(t)]$.

Proof. Using the definition of the Laplace transform

$$\begin{aligned} L[tf(t)] &= \int_0^\infty tf(t)e^{-st}dt \\ &= -\int_0^\infty \frac{\partial}{\partial s}\left[f(t)e^{-st}\right]dt \\ &= -\frac{d}{ds}\int_0^\infty f(t)e^{-st}dt \\ &= -\frac{d}{ds}F(s), \end{aligned}$$

where we have assumed that the interchange of differentiation and integration is possible. Replacing f by tf, we get

$$L[t^2 f(t)] = -\frac{d}{ds}L[tf(t)] = (-1)^2 \frac{d^2}{ds^2}F(s).$$

Proceeding in this way, we obtain

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s).$$

□

Example 1.10. Find the Laplace transform of $t^2 \sin 3t$.

Solution: We have $f(t) = \sin 3t$. So $F(s) = \frac{3}{s^2+9}$. Using the results of above theorem, we get

$$L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}\frac{3}{s^2+9} = \frac{6s^2 - 54}{(s^2+9)^3}.$$

Theorem 1.6. ***(Integration of the Laplace transform)****Let f be a piecewise continuous function $[0,\infty)$ and be of exponential order. If $\lim_{t\to 0}\frac{f(t)}{t}$ exists, then*

$$L\left[\frac{f(t)}{t}\right]=\int_s^\infty F(\tau)d\tau, \tag{1.4}$$

where $F(s)=L[f(t)]$.

Proof. By definition, we have

$$\begin{aligned}L\left[\frac{f(t)}{t}\right]&=\int_0^\infty \frac{f(t)}{t}e^{-st}dt\\&=\int_0^\infty f(t)\int_s^\infty e^{-\tau t}d\tau dt\\&=\int_s^\infty\int_0^\infty e^{-\tau t}f(t)dtd\tau\\&=\int_s^\infty F(\tau)d\tau,\end{aligned}$$

where we have assumed that interchange of the order of integration is possible. □

Example 1.11. Find the Laplace transform of $\frac{\sin wt}{t}$.

Solution: Let $f(t)=\sin wt$. Then $F(s)=\frac{w}{s^2+w^2}$. By the relation (1.4), we get

$$\begin{aligned}L\left[\frac{f(t)}{t}\right]&=\int_s^\infty \frac{w}{\tau^2+w^2}d\tau\\&=\left[\tan^{-1}(\frac{\tau}{w})\right]_s^\infty\\&=\frac{\pi}{2}-\tan^{-1}(\frac{s}{w}).\end{aligned}$$

1.5. **Questions.**

(1) Find the Laplace transform of the following functions
   (a) $\frac{1}{(s-a)^2}$.
   (b) $\frac{1}{(s^2-a^2)^2}$.
   (c) $\frac{2s+3}{(s^2+2s+2)^2}$.
   (d) $\frac{s}{(s^2+a^2)^2}$.
(2) Find the solution of the following problems
   (a) $ty'-3y=2t, y(0)=1$.
   (b) $y''+6ty'-12y=1,\ y(0)=2,\ y'(0)=1$.
   (c) $y''+ty'-2y=6-t,\ y(0)=0,\ y'(0)=1$.
   (d) $ty''+4ty'-12y=0,\ y(0)=0,\ y'(0)=-2$.

1.6. **Shifting Theorems and Dirac Delta Function.** The shifting theorems are useful to obtain the Laplace transform as well as inverse Laplace transform of some functions. If we multiply f by e^{at} in t variable resulted a shifting by a in s variable. Similarly shifting by a in t variable results e^{-as} multiplication in s variable. We now establish the following shifting theorem.

Theorem 1.7. ***(First shifting theorem)*** *Let* $L[f(t)] = F(s), s > \alpha \geq 0$ *and* a *be any real number. Then*

$$L[e^{at}f(t)] = F(s-a),\ s > a + \alpha. \tag{1.5}$$

Proof. By definition of the Laplace transform

$$\begin{aligned} L[e^{at}f(t)] &= \int_0^\infty e^{at}f(t)e^{-st}dt \\ &= \int_0^\infty e^{-(s-a)t}f(t) \\ &= F(s-a),\ s > a + \alpha. \end{aligned}$$

□

Definition 1.4. The Heaviside function or the unit step function denoted by H or U, is defined as

$$H(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geq 0. \end{cases}$$

From the definition of the Heaviside function, it is clear that $t = 0$ is a point of discontinuity of H. The Heaviside function is helpful in solving the differential equations when the forcing term has points of jump discontinuity. If the point of discontinuity is shifted to the point $t = a$, then we have the following definition for the Heaviside function

$$H(t-a) = \begin{cases} 0, & \text{if } t < a \\ 1, & \text{if } t \geq a. \end{cases}$$

Theorem 1.8. ***(Second shifting theorem)*** *Let* $L[f(t)] = F(s), s > \alpha \geq 0$ *and* $a \geq 0$. *Then*

$$L[f(t-a)U(t-a)] = e^{-as}F(s). \tag{1.6}$$

Proof. By the definition of Laplace transform, we get

$$\begin{aligned} L[f(t-a)U(t-a)] &= \int_0^\infty f(t-a)U(t-a)e^{-st}dt \\ &= \int_a^\infty f(t-a)e^{-st}dt \text{ (put } \tau = t-a,\ d\tau = dt) \\ &= \int_0^\infty f(\tau)e^{-(\tau+a)s}d\tau \\ &= e^{-as}\int_0^\infty f(\tau)e^{-\tau s}d\tau \\ &= e^{-as}F(s). \end{aligned}$$

□

Definition 1.5. The Dirac-delta function is defined as

$$\delta(t) = \lim_{k\to 0+} \frac{H(t) - H(t-k)}{k}, \tag{1.7}$$

where $H(t)$ is the Heaviside function.

1.7. **Properties of Dirac-Delta Function.** We recall the definition of the Dirac-delta function

$$\delta(t) = \lim_{\epsilon\to 0} \delta_\epsilon(t) = \lim_{\epsilon\to 0+} \frac{H(t+\epsilon) - H(t-\epsilon)}{2\epsilon}.$$

Property I: For an integrable function we have

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0).$$

Proof. From the definition, we have

$$\int_{-\infty}^{\infty} \delta_\epsilon(t)dt = \int_{-\epsilon}^{\epsilon} \frac{1}{2\epsilon}dt = 1. \tag{1.8}$$

If f is integrable function on $(-\epsilon, \epsilon)$, then it follows from the mean value theorem of integral equation that

$$\int_{-\infty}^{\infty} f(t)\delta_\epsilon(t)dt = \int_{-\epsilon}^{\epsilon} \frac{1}{2\epsilon}f(t)dt = f(c), \text{ for some } c \in (-\epsilon, \epsilon). \tag{1.9}$$

Taking limit as $\epsilon \to 0$, we obtain

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0).$$

□

Property II: If f is a continuous function on $(a, a+\epsilon)$ for $\epsilon > 0$, then we have

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a).$$

Proof. We define the Dirac-delta function as

$$\delta_\epsilon(t-a) = \begin{cases} \frac{1}{\epsilon}, & a < t < a+\epsilon \\ 0, & \text{elsewhere.} \end{cases}$$

$$\begin{aligned} \int_{-\infty}^{\infty} f(t)\delta_\epsilon(t-a)dt &= \frac{1}{\epsilon}\int_0^{a+\epsilon} f(t)dt \\ &= f(a+\theta\epsilon),\ 0 < \theta < 1. \end{aligned}$$

Taking limit as $\epsilon \to \infty$, we obtain

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a).$$

□

Remark 1.4. It follows from the definition of δ that

$$\delta(t) = \begin{cases} 0, & \text{if } t \neq 0 \\ 1, & \text{if } t = 0. \end{cases}$$

Remark 1.5. The definition of H implies that

$$H'(t) = \delta(t).$$

Taking the Laplace transform to $\frac{H(t-a)-H(t-a-k)}{k}$, we get

$$\begin{aligned} L\left[\frac{H(t-a)-H(t-a-k)}{k}\right] &= \frac{1}{k}\left[\frac{e^{-as}}{s} - \frac{e^{-(a+k)s}}{s}\right] \\ &= e^{-as}\frac{1-e^{-ks}}{ks}. \end{aligned}$$

Taking limit as $k \to 0+$, we get

$$L[\delta(t-a)] = e^{-as}$$

as $\frac{1-e^{-ks}}{ks} \to 1$. Putting $a = 0$, $L[\delta(t)] = 1$.

Example 1.12. Solve $y'' + 2y' + 10y = 6\delta(t-2) - 3\delta(t-3),\ y(0) = 0,\ y'(0) = 0.$

Solution: Taking the Laplace transform to both sides of the given equation, we get

$$\begin{aligned} s^2L[y] + 2L[y] + 10L[y] &= 6e^{-2s} - 3e^{-3s} \\ \Rightarrow L[y] &= \frac{6e^{-2s}}{s^2+2s+10} - \frac{3e^{-3s}}{s^2+2s+10} \\ &= \frac{6e^{-2s}}{(s+1)^2+3^2} - \frac{3e^{-3s}}{(s+1)^2+3^2}. \end{aligned}$$

Taking the inverse Laplace transform, we get

$$y(t) = \frac{6}{3}H(t-2)e^{-(t-2)}\sin 3(t-2) - \frac{3}{3}H(t-3)e^{-(t-3)}\sin 3(t-3).$$

1.8. **Questions.**

(1) Prove that $H'(t) = \delta(t)$.

(2) Find the value of $\int_0^\infty f(t)\delta(t-1)dt$, where

$$f(t) = \begin{cases} t^2 & ,0 \le t < 1, \\ 2, & t = 1, \\ t, & t > 1. \end{cases}$$

(3) Solve the following initial value problem

(a) $y'' + 4y + 5y = \delta(t-3),\ y(0) = 0,\ y'(0) = 0.$

(b) $y'' + 9y = 4\delta(t),\ y(0) = 0,\ y'(0) = 0.$

(c) $y'' + 4y' + 8y = 16\delta(t-1) + 8\delta(t-2),\ y(0) = 0,\ y'(0) = 0.$

1.9. **Convolution Theorem.** The convolution is useful in obtaining the inverse Laplace transform of product two functions in s variable. The inverse is obtained in terms of convolution which is defined in terms of integration.

Definition 1.6. Let f and g be two functions defined on $[0, \infty)$. Then the *convolution* of f and g is defined as

$$f * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau \tag{1.10}$$

provided the integral exists.

The following properties can be verified by using the definition of the convolution.

(i) $f * g = g * f$.

(ii) $f * (h_1 + h_2) = f * h_1 + f * h_2$.

(iii) $f * (g * h) = (f * g) * h$.

(iv) $f * 0 = 0$.

We note that $1 * f \neq f$. For example we can take $f(t) = k$ for some constant k. Then

$$\begin{aligned} 1 * k(t) &= \int_0^t 1.kd\tau \\ &= kt \neq k \text{ for any } \ t \geq 0. \end{aligned}$$

We now give the Laplace transform of convolution of two functions which is the point-wise product of the Laplace transform of the functions.

Theorem 1.9. *Let f and g be piecewise continuous functions on $[0,\infty)$ and be of exponential orders. Then*

$$L[f*g(t)] = L[f(t)]L[g(t)]. \tag{1.11}$$

Proof. By the definition of the Laplace transform, we have

$$\begin{aligned}
L[f*g(t)] &= \int_0^\infty f*g(t)e^{-st}dt \\
&= \int_{t=0}^\infty \int_{\tau=0}^t f(\tau)g(t-\tau)d\tau e^{-st}dt \\
&= \int_{\tau=0}^\infty \int_{t=\tau}^\infty f(\tau)g(t-\tau)e^{-st}dtd\tau \text{ (changing the order of integration)} \\
&= \int_{\tau=0}^\infty \int_{u=0}^\infty f(\tau)g(u)e^{-s(\tau+u)}dud\tau \text{ (put } t-\tau=u,\ dt=du) \\
&= \int_{\tau=0}^\infty f(\tau)e^{-s\tau}d\tau \int_{u=0}^\infty g(u)e^{-su}du \\
&= L[f(t)]L[g(t)] \\
&= F(s)G(s),
\end{aligned}$$

where $L[f(t)] = F(s), L[g(t)] = G(s)$. □

Example 1.13. Find the inverse Laplace transform of $\frac{18}{(s^2+9)^2}$.

Solution: We have

$$\begin{aligned}
\frac{18}{(s^2+9)^2} &= 2\frac{3}{s^2+9}\frac{3}{s^2+9} \\
&= F(s)G(s),
\end{aligned}$$

where $F(s) = 2\frac{3}{s^2+9}$ and $G(s) = \frac{3}{s^2+9}$. Then $f(t) = L^{-1}[F(s)] = 2\sin 3t$ and $g(t) = L^{-1}[G(s)] = \sin 3t$. Thus

$$\begin{aligned}
L^{-1}\left[\frac{18}{(s^2+9)^2}\right] &= L^{-1}\left[2\frac{3}{s^2+9}\frac{3}{s^2+9}\right] \\
&= \int_0^t 2\sin 3\tau \sin 3(t-\tau)d\tau \\
&= \int_0^t [\cos 3(t-2\tau) - \cos 3t]d\tau \\
&= \left[\frac{\sin 3(t-2\tau)}{-6} - t\cos 3t\right]_0^t \\
&= \frac{1}{3}[\sin 3t - 3t\cos 3t].
\end{aligned}$$

Example 1.14. Solve the integral equations

$$f(t) = 1 + t + 2\int_0^t \sin\tau f(t-\tau)d\tau.$$

Solution: Taking the Laplace transform to the given equation, we get

$$\begin{aligned} L[f(t)] &= \frac{1}{s} + \frac{1}{s^2} + 2L[\sin t * f(t)] \\ &= \frac{1}{s} + \frac{1}{s^2} + 2\frac{1}{s^2+1}L[f(t)] \\ \Rightarrow L[f(t)](1 - 2\frac{1}{s^2+1}) &= \frac{1}{s} + \frac{1}{s^2} \\ \Rightarrow L[f(t)] &= \left(\frac{1}{s} + \frac{1}{s^2}\right)\frac{s^2+1}{s^2-1} \\ &= \frac{2}{s-1} - \frac{1}{s^2} - \frac{1}{s}. \end{aligned}$$

Taking the inverse Laplace transform, we get

$$f(t) = 2e^t - t - 1.$$

1.10. **Questions.**

(1) Give an example to show that $f * f$ may not be always nonnegative.
(2) Use convolution theorem to find the inverse of the following
   (a) $\frac{1}{(s-a)(s-b)^2}$.
   (b) $\frac{1}{(s^2-a^2)(s^2-b^2)^2}$.
   (c) $\frac{8s}{(s^2+16)(s^2+1)^2}$.
   (d) $\frac{1}{(s^2+4)(s^2-16)}$.
(3) Use convolution theorem to solve the following initial value problems
   (a) $y'' + 4y' + 4y = te^t,\ y(0) = 0,\ y'(0) = 2$.
   (b) $y'' - w^2y = \cosh wt,\ y(0) = 1,\ y'(0) = 2$.
   (c) $y'' - 5y' - 6y = e^{-t},\ y(0) = 1,\ y'(0) = 1$.
   (d) $y' - y = te^t\cos t,\ y(0) = 0$.
(4) Use convolution theorem to solve the following integral equations
   (a) $f(t) = 1 + t + 6\int_0^t f(\tau)e^{t-\tau}d\tau$.
   (b) $f(t) = e^t + \int_0^t \tau f(t-\tau)d\tau$.
   (c) $f(t) = \cos t + e^{-t}\int_0^t e^{\tau}f(\tau)d\tau$.
   (d) $f(t) = t - \int_0^t (e^{\tau} + e^{-\tau})f(t-\tau)d\tau$.

1.11. **Laplace Transform of Periodic Functions.** Periodic functions are common in many applications in science and engineering. To obtain the solution of some initial value problems with periodic force term, we need to find the Laplace transform of periodic functions.

Definition 1.7. A function f on $[0,\infty)$ is said to be periodic of period T if

$$f(t+T)=f(t),\ t>0.$$

Theorem 1.10. *Let f be a piecewise continuous periodic function on $[0,\infty)$ with period T and be of exponential order α. Then*

$$L[f(t)]=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt,\ s>0. \tag{1.12}$$

Proof. By the definition of the Laplace transform, we have

$$\begin{aligned}
&L[f(t)]\\
&=\int_0^\infty e^{-st}f(t)dt\\
&=\int_0^T e^{-st}f(t)dt+\int_T^{2T} e^{-st}f(t)dt+\int_{2T}^{3T} e^{-st}f(t)dt+...\\
&=\int_0^T e^{-st}f(t)dt+\int_0^T e^{-s(u+T)}f(u+T)du+\int_0^T e^{-s(u+2T)}f(u+2T)du+...,
\end{aligned}$$

where we have put $t=u+T$, $t=u+2T$,... in the second, third integral etc. on the right hand side of the last equation. Since f is periodic of period T, so we get

$$\begin{aligned}
L[f(t)]&=\int_0^T e^{-st}f(t)dt+\int_0^T e^{-s(u+T)}f(u+T)du+\int_0^T e^{-s(u+2T)}f(u+2T)du+...\\
&=\int_0^T e^{-st}f(t)dt+e^{-sT}\int_0^T e^{-su}f(u)du+e^{-2sT}\int_0^T e^{-su}f(u)du+...\\
&=(1+e^{-sT}+e^{-2sT}+...)\int_0^T e^{-st}f(t)dt\\
&=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt.
\end{aligned}$$

□

Example 1.15. Find the Laplace transform of the periodic function

$$f(t)=t,\ 0\le t\le a,\ f(t+a)=f(t).$$

Solution: Here $T = a$. Then by (1.12), we get

$$\begin{aligned}
&L[f(t)] \\
&= \frac{1}{1-e^{-sa}} \int_0^a e^{-st} f(t)dt \\
&= \frac{1}{1-e^{-sa}} \int_0^a e^{-st} t dt \\
&= \frac{1}{1-e^{-sa}} \big[-\big(\frac{t}{s} + \frac{1}{s^2}\big) e^{-st} \big]_0^a \\
&= \frac{1}{1-e^{-sa}} \big[-\big(\frac{a}{s} + \frac{1}{s^2}\big) e^{-sa} + \frac{1}{s^2} \big].
\end{aligned}$$

Example 1.16. Solve $y'' + 4y' + 5y = f(t),\ y(0) = 0,\ y'(0) = 0$ and

$$f(t) = \begin{cases} 1, & 0 \le t < \pi, \\ -1, & \pi \le t \le 2\pi. \end{cases}$$

Solution: The given function has period 2π. Taking the Laplace transform to the given equation and using the initial value, we get

$$\begin{aligned}
s^2 L[y] + 4sL[y] + 5L[y] &= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)dt \\
&= \frac{1}{1-e^{-2\pi s}} \left(\int_0^{\pi} e^{-st} dt + \int_{\pi}^{2\pi} e^{-st}(-1)dt \right) \\
&= \frac{1}{1-e^{-2\pi s}} \left(\frac{1-e^{-s\pi}}{s} + \frac{e^{-2s\pi} - e^{-s\pi}}{s} \right) \\
&= \frac{1-e^{-\pi s}}{s(1+e^{-\pi s})} \\
&= \frac{2}{s(1+e^{-\pi s})} - \frac{1}{s} \\
\Rightarrow L[y] &= \frac{2}{s(1+e^{-\pi s})(s^2+4s+5)} - \frac{1}{s(s^2+4s+5)} \\
&= \frac{2}{s(s^2+4s+5)} \big(1 - e^{-\pi s} + e^{-2\pi s} - e^{-3\pi s} + ...\big) \\
&- \frac{1}{s(s^2+4s+5)}.
\end{aligned}$$

We have

$$L^{-1}\left[\frac{1}{s(s^2+4s+5)} \right] = \int_0^t e^{-2t} \sin t dt = \frac{1}{5} - \frac{1}{5}(2\sin t + \cos t) = \frac{1}{5} - g(t),$$

where $g(t) = \frac{1}{5}(2\sin t + \cos t)$. Using this we get

$$y(t) = \left[\frac{1}{5} - g(t)\right]H(t) - 2\left[\frac{1}{5} - g(t-\pi)\right]H(t-\pi)$$
$$+ 2\left[\frac{1}{5} - g(t-2\pi)\right]H(t-2\pi) + ...$$

1.12. **Questions.**

(1) Find the Laplace transform of the following periodic functions with one period defined as

(a) $f(t) = t^2,\ 0 < t < 2\pi.$

(b) $f(t) = \begin{cases} t, & 0 < t < a, \\ 0, & a < t < 2a. \end{cases}$

(c) $f(t) = \begin{cases} t, & 0 < t \leq \pi, \\ 2\pi - t, & \pi < t < 2\pi. \end{cases}$

(d) $f(t) = \begin{cases} \cos t, & 0 < t \leq \pi/2, \\ -\cos t, & \pi/2 < t < 3\pi/2, \\ \cos t, & 3\pi/2 < t < 2\pi. \end{cases}$

(e) Solve the following initial value problems

(f) $y'' + 4y = f(t),\ y(0) = 0,\ y'(0) = 0$ and $f(t) = \begin{cases} \sin t, & 0 \leq t < \pi, \\ 0, & \pi \leq t < 2\pi. \end{cases}$

(g) $y'' + y = f(t),\ y(0) = 0,\ y'(0) = 0$ and $f(t) = \begin{cases} 0, & 0 \leq t < \pi, \\ \sin t, & \pi \leq t < 2\pi. \end{cases}$

(h) $y'' + 6y' + 10y = f(t),\ y(0) = 0,\ y'(0) = 1$ and $f(t) = \begin{cases} 1, & 0 \leq t < \pi, \\ 0, & \pi \leq t < 2\pi. \end{cases}$

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